CONSTANT COEFFICIENT LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We build the theory of linear ordinary differential equations with constant coefficients, by using purely linear methods (that is, without relying on any general arguments such as the Picard-Lindelöf theorem for existence and the energy method for uniqueness). Constant coefficient first order linear systems are also treated with similar methods.

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1. First order homogeneous equations

Consider the differential equation

$$u' + pu = 0, \tag{1}$$

where p is a given constant. We assume that the constant p and the unknown function $u(t)$ are *complex valued* by default, because the complex case contains the real case, and will indicate whenever we want to require everything to be real valued.

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The equation (1) can easily be solved by looking for the solution in the form $u(x) = e^{\lambda t}$, and finding that $\lambda = -p$. Then linearity (or equivalently the superposition principle) implies that

$$u(t) = Ae^{-pt}, \tag{2}$$

is a solution for any constant $A \in \mathbb{C}$. However, such a method would give us uniqueness only in a restrictive class; namely, we would not be able to rule out other possible solutions that are *not* of the form $e^{\lambda t}$.

The following method cures the defficiency. Suppose that u is a differentiable function on $\mathbb{R}$, satisfying (1) everywhere in $\mathbb{R}$. Then the new function v defined by

$$v(t) = e^{pt}u(t), \qquad t \in \mathbb{R}, \tag{3}$$

satisfies

$$v'(t) = pe^{pt}u(t) + e^{pt}u'(t) = e^{pt}(pu(t) + u'(t)) = 0, \tag{4}$$

for all $t \in \mathbb{R}$. Since v is differentiable and its derivative vanishes everywhere, (as a consequence of the mean value theorem) we conclude that v is equal to a constant. Let us state this important result as a theorem.

Theorem 1. *The only function that is differentiable everywhere and satisfies* (1) *everywhere in* $\mathbb{R}$ *is of the form* $u(t) = Ae^{-pt}$*, with an arbitrary constant* $A \in \mathbb{C}$*.*

We say that the solution space of (1) is parametrized by a single complex number.

Exercise 2. This theorem remains true if we replace $\mathbb{R}$ by some open interval $(a, b) \subset \mathbb{R}$.

Let us now consider the case where the coefficient p is *real*. In this case, the space of *real solutions* of (1) is parametrized by a single *real* number.

Corollary 3. *If* $p \in \mathbb{R}$*, then the only real valued function that is differentiable everywhere and satisfies* (1) *everywhere in* $\mathbb{R}$ *is of the form* $u(t) = Ae^{-pt}$*, with an arbitrary constant* $A \in \mathbb{R}$*.*

Proof. By Theorem 1, any solution of (1) must be of the form $u(t) = Ae^{-pt}$. If $u(t)$ is real, then $A = u(t)e^{pt}$ must be real. On the other hand, if A is real, then $u(t) = Ae^{-pt}$ is real. The proof is established. □

Exercise 4. Let u be a solution to $u' + pu = 0$ with p real.

(a) Show that both real and imaginary parts of u are solutions.
(b) Show that if $u(0)$ is real, then $u(t)$ is real for all t.

2. Stability

It is now immediate that the only solution to the inital value problem

$$\begin{cases} u' = \ell u \quad \text{in } \mathbb{R}, \\ u(0) = \alpha, \end{cases} \tag{5}$$

where $\alpha, \ell \in \mathbb{C}$ are given constants, is

$$u(t) = \alpha e^{\ell t}. \tag{6}$$

By a solution, we mean here a function u that is differentiable everywhere and satisfies (5).

Remark 5. The *stability behaviour* of this solution for real or purely imaginary ℓ can be summarized as follows.

- If $\ell > 0$, then all solutions (except for the case $\alpha = 0$) grow unboundedly as $t \to \infty$. This is an example of *instability*.
- If $\ell < 0$, then all solutions decay to 0 as $t \to \infty$. This system is *strictly stable*.

- If $\ell = 0$ or more generally $\ell = i\omega$ with $\omega \in \mathbb{R}$, then the solution stays bounded as $t \to \infty$. We say that the system is *stable*.

Definition 6. An ODE is called *strictly stable* if all of its solutions decay to 0 as $t \to \infty$. It is called *stable* if all of its solutions stay bounded as $t \to \infty$, that is, for any solution u, there exists a constant M such that $|u(t)| \leq M$ for all $t > 0$. If an ODE is not stable, then we say that it is *unstable*.

Exercise 7. Consider the equation $u' = \ell u$, where $\ell = \delta + i\omega$ is a constant with δ and ω real. Identify the stable, unstable, and strictly stable regimes.

3. Duhamel's formula

In this section, we consider the *inhomogenous* initial value problem

$$\begin{cases} u' = \ell u + f \quad \text{in } (0,T), \\ u(0) = \alpha, \end{cases} \tag{7}$$

where the constants $0 < T \leq \infty$ and $\alpha, \ell \in \mathbb{C}$, and the function $f \in C[0,T)$ are given. Here the notation $f \in C[0,T)$ means that f is continuous on the interval $[0,T)$. Suppose that $u \in C[0,T)$ and that u is differentiable in $(0,T)$. Then the function

$$v(t) = e^{-\ell t}u(t), \qquad t \in [0,T), \tag{8}$$

satisfies

$$v'(t) = -\ell e^{-\ell t}u(t) + e^{-\ell t}u'(t) = e^{-\ell t}(-\ell u(t) + u'(t)) = e^{-\ell t}f(t), \tag{9}$$

for $t \in (0,T)$, and the fundamental theorem of calculus yields

$$\int_0^t e^{-\ell s}f(s)ds = v(t) - v(0) = e^{-\ell t}u(t) - u(0), \tag{10}$$

for $t \in [0,T)$, or

$$u(t) = e^{\ell t}\alpha + \int_0^t e^{\ell(t-s)}f(s)ds, \qquad t \in [0,T). \tag{11}$$

This is called *Duhamel's formula*, or *variation of constants formula*.

Exercise 8. Show that (11) is a solution of the initial value problem (7).

Theorem 9. *Let $f \in C[0,T)$ with $0 < T \leq \infty$. Then the only solution of (7) that is continuous on $[0,T)$ and differentiable in $(0,T)$ is given by (11).*

Exercise 10. Prove the following "shifted" version of Duhamel's formula. Let $f \in C[a,b)$ with $-\infty < a < b \leq \infty$. Then the only solution of

$$u' = \ell u + f \quad \text{in } [a,b), \tag{12}$$

that is continuous on $[a,b)$ and differentiable in (a,b) is given by

$$u(t) = e^{\ell(t-a)}u(a) + \int_a^t e^{\ell(t-s)}f(s)ds, \qquad t \in (a,b). \tag{13}$$

Exercise 11. Prove the following "reflected" (or "backward in time") version of Duhamel's formula. Let $f \in C(a,b]$ with $-\infty \leq a < b < \infty$. Then the only solution of

$$u' = \ell u + f \quad \text{in } (a,b), \tag{14}$$

that is continuous on $(a,b]$ and differentiable in (a,b) is given by

$$u(t) = e^{\ell(t-b)}u(b) - \int_t^b e^{\ell(t-s)}f(s)ds, \qquad t \in (a,b]. \tag{15}$$

Note that this could be subsumed under (13) if we agree that $\int_a^b g dt = -\int_b^a g dt$ when $b < a$.

4. Second order homogeneous equations

Now consider the second order equation

$$u'' + pu' + qu = 0, \tag{16}$$

where p and q are (real or complex) constants. Substituting the form $u(x) = e^{\lambda t}$ leads to the so-called *characteristic equation*

$$\lambda^2 + p\lambda + q = 0. \tag{17}$$

With λ_1 and λ_2 denoting the roots of the characteristic equation (which we call *characteristic roots*), we get two solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ (which obviously collapses into one if $\lambda_1 = \lambda_2$). Linearity (or the superposition principle) then says that

$$u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \tag{18}$$

is a solution for arbitrary values of the constants $A, B \in \mathbb{C}$. If $\lambda_1 \neq \lambda_2$, this is a 2-parameter family of solutions. On the other hand, if $\lambda_1 = \lambda_2$, one can check that $te^{\lambda_1 t}$ is also a solution, so that a 2-parameter family of solutions is given by

$$u(t) = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}. \tag{19}$$

Is there any solution that is not of the form (18) or (19)? The following theorem says no.

Theorem 12. *The only function that is twice differentiable everywhere and satisfies* (16) *everywhere in* $\mathbb{R}$ *is of the form either* (19) *or* (18) *with arbitrary constants* $A, B \in \mathbb{C}$*, depending on whether or not the characteristic equation* (17) *has repeated roots.*

Proof. Suppose that u is twice differentiable everywhere and satisfies (16) everywhere in $\mathbb{R}$. Let λ_1 and λ_2 be the roots of the characteristic equation (17), and let

$$v(t) = u'(t) - \lambda_1 u(t), \qquad t \in \mathbb{R}. \tag{20}$$

We then compute

$$v' - \lambda_2 v = u'' - \lambda_1 u' - \lambda_2(u' - \lambda_1 u) = u'' - (\lambda_1 + \lambda_2)u' + \lambda_1\lambda_2 u = 0, \tag{21}$$

everywhere in $\mathbb{R}$ because $\lambda_1 + \lambda_2 = -p$ and $\lambda_1\lambda_2 = q$, meaning that

$$v(t) = v(0)e^{\lambda_2 t}, \qquad t \in \mathbb{R}. \tag{22}$$

On account of (20), Duhamel's formula (11) yields

$$u(t) = u(0)e^{\lambda_1 t} + v(0)\int_0^t e^{\lambda_1(t-s)}e^{\lambda_2 s}ds. \tag{23}$$

If $\lambda_1 = \lambda_2$, we have

$$u(t) = u(0)e^{\lambda_1 t} + v(0)te^{\lambda_1 t}, \tag{24}$$

which is of the form (18), and if $\lambda_1 \neq \lambda_2$, we have

$$u(t) = u(0)e^{\lambda_1 t} + v(0)e^{\lambda_1 t}\cdot\frac{e^{(\lambda_2-\lambda_1)t} - 1}{\lambda_2 - \lambda_1} = \big(u(0) - \frac{v(0)}{\lambda_2 - \lambda_1}\big)e^{\lambda_1 t} + \frac{v(0)e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \tag{25}$$

which is of the form (19). □

Thus the solution space of (16) is parametrized by 2 complex numbers.

Exercise 13. This theorem remains true if we replace $\mathbb{R}$ by some open interval $(a, b) \subset \mathbb{R}$.

Remark 14. From now on, by a solution of (16) we will always understand a function that is twice differentiable everywhere and satisfies (16) everywhere in $\mathbb{R}$ (or in some specified interval). Note that the formula (23) gives the representation

$$u(t) = u(0)e^{\lambda_1 t} + [u'(0) - \lambda_1 u(0)]e^{\lambda_1 t}\int_0^t e^{(\lambda_2-\lambda_1)s}ds, \tag{26}$$

for any solution u of (16).

Now we focus on the real coefficients case $p, q \in \mathbb{R}$. It turns out that the real solutions of (16) are parametrized by 2 real numbers, or equivalently, by a single complex number.

Corollary 15. *Let p and q be real. Then the only real valued function that is twice differentiable everywhere and satisfies (16) everywhere in $\mathbb{R}$ is in one of the following forms.*

- $u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ *with arbitrary constants $A, B \in \mathbb{R}$, if $\lambda_1 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$.*
- $u(t) = (A + Bt)e^{\lambda_1 t}$ *with arbitrary constants $A, B \in \mathbb{R}$, if $\lambda_1 = \lambda_2$.*
- $u(t) = \mathrm{Re}(Ae^{\lambda_1 t})$ *with an arbitrary constant $A \in \mathbb{C}$, if $\lambda_1 \notin \mathbb{R}$.*

The third form can also be written as

$$u(t) = e^{\mu t}(E\cos\omega t + F\sin\omega t), \tag{27}$$

with arbitrary constants $E, F \in \mathbb{R}$, where $\mu = \mathrm{Re}(\lambda_1)$ and $\omega = \mathrm{Im}(\lambda_1)$.

Proof. Since $p, q \in \mathbb{R}$, the characteristic roots are either real or complex conjugates of each other. If the roots are real and distinct, then taking A and B real, the formula (18) produces real solutions $u(t)$. On the other hand, let A and B be not both real. Then $u(0) = A + B$ is not real unless $A = \bar{B}$. If $A = \bar{B}$ then $u(1) = e^{\lambda_1}(A + Be^{\lambda_2-\lambda_1})$ would not be real. In any case, $u(t)$ is real for all t if and only if both A and B are real.

Now suppose that $\lambda_1 = \lambda_2$, which can only happen if $\lambda_1 \in \mathbb{R}$. Then (19) produces real solutions $u(t)$ for A and B real. On the other hand, if A and B are not both real, then $u(0) = A$ or $u(1) = (A + B)e^{\lambda_1}$ would not be real.

Finally, if the roots are not real, then they must be of the form $\lambda_{1,2} = \mu \pm \omega i$ for some real μ and $\omega \neq 0$, meaning that (18) becomes

$$\begin{aligned} u(t) &= Ae^{\mu t}e^{i\omega t} + Be^{\mu t}e^{-i\omega t} = Ae^{\mu t}(\cos\omega t + i\sin\omega t) + Be^{\mu t}(\cos\omega t - i\sin\omega t) \\ &= e^{\mu t}(A + B)\cos\omega t + e^{\mu t}i(A - B)\sin\omega t. \end{aligned} \tag{28}$$

Therefore, if we take both $A + B$ and $i(A - B)$ to be real, that is, if we take $A = a - bi$ and $B = a + bi = \bar{A}$ for some real a and b, then the solution

$$u(t) = 2e^{\mu t}(a\cos\omega t + b\sin\omega t) = Ae^{\lambda_1 t} + \bar{A}e^{\bar{\lambda}_1 t} = \mathrm{Re}(2Ae^{\lambda_1 t}), \tag{29}$$

is real. On the other hand, if $B \neq \bar{A}$ in (18), then $u(0) = A + B$ would not be real. The proof is established. □

Exercise 16. Let u be a solution to $u'' + pu' + qu = 0$ with p and q real.
(a) Show that both real and imaginary parts of u are solutions.
(b) Show that if $u(0)$ and $u'(0)$ are real, then $u(t)$ is real for all t.

5. Stability

Let λ_1 and λ_2 be the characteristic roots corresponding to the equation

$$u'' + pu' + qu = 0. \tag{30}$$

Its stability properties are fully determined in the following theorem.

Theorem 17. *The system (30) is strictly stable if and only if $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) < 0$. It is stable if and only if $\mathrm{Re}(\lambda_1) \leq 0$ and $\mathrm{Re}\lambda_2) \leq 0$, with $\lambda_1 \neq \lambda_2$ whenever $\mathrm{Re}(\lambda_1) = 0$.*

Proof. Any function of the form $t^k e^{\lambda t}$ with $\mathrm{Re}(\lambda) < 0$ decays to 0 as $t \to \infty$, and $e^{\lambda t}$ with $\mathrm{Re}(\lambda) \geq 0$ does not decay as $t \to \infty$. Thus the first claim of the theorem is established.

Similarly, any function of the form $t^k e^{\lambda t}$ with $\mathrm{Re}(\lambda) < 0$ or of the form $e^{\lambda t}$ with $\mathrm{Re}(\lambda) = 0$ stays bounded as $t \to \infty$. On the other hand, neither $e^{\lambda t}$ with $\mathrm{Re}(\lambda) > 0$ nor $te^{\lambda t}$ with $\mathrm{Re}(\lambda) = 0$ stays bounded as $t \to \infty$. The proof is complete. □

Let us now look at the case where p and q are real numbers. This is the subclass of the equation (16) with real coefficients. In particular, if $q > 0$ and $p \geq 0$, it describes a damped harmonic oscillator with the natural frequency $\omega = \sqrt{q}$.

As the characteristc roots are $\lambda_{1,2} = (-p \pm \sqrt{p^2 - 4q})/2$, the first observation we can make is that all nontrivial solutions are oscillatory if and only if $p^2 < 4q$.

Corollary 18. *The system (30) with $p, q \in \mathbb{R}$ is strictly stable if and only if $p > 0$ and $q > 0$. It is stable if and only if $p \geq 0$ and $q \geq 0$ with $p + q > 0$,*

Proof. Recall that the system is strictly stable if and only if $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) < 0$. This happens if and only if $p > 0$ and either $\sqrt{p^2 - 4q} < p$ or $\sqrt{p^2 - 4q}$ is purely imaginary. The latter "either or" condition is precisely equivalent to $q > 0$. Thus the first claim of the theorem is established.

For the second claim, we start with the "only if" part. The case $p = q = 0$ is unstable as $\lambda_1 = \lambda_2 = 0$. If $q < 0$ then $\lambda_1 = (-p + \sqrt{p^2 - 4q})/2 > 0$. If $p < 0$ then $\mathrm{Re}(\lambda_1) > 0$.

What remains is the "if" part of the second claim. Since the case $p > 0$ and $q > 0$ is strictly stable, we only need to treat the borderline cases. If $p > 0$ and $q = 0$ then $\lambda_1 = -p$ and $\lambda_2 = 0$. If $p = 0$ and $q > 0$ then $\lambda_{1,2} = \pm i\sqrt{q}$. The proof is complete. □

6. Initial value problems

As the general solution of a second order ODE involves *two* arbitrary constants, the corresponding initial value problem has the form

$$\begin{cases} u'' + pu' + qu = 0, \\ u(0) = \alpha, \\ u'(0) = \beta. \end{cases} \tag{31}$$

If we write the general solution of the differential equation as

$$u(t) = A_1\phi_1(t) + A_2\phi_2(t), \tag{32}$$

with some fixed solutions ϕ_1 and ϕ_2, the coefficients A_1 and A_2 should be determined from the initial conditions $u(0) = \alpha$ and $u'(0) = \beta$.

Example 19. As the characteristic roots of the equation

$$u'' + u = 0, \tag{33}$$

are $\lambda_{1,2} = \pm i$, the general solution is

$$u(t) = Ae^{it} + Be^{-it}. \tag{34}$$

This can also be written as

$$u(t) = A(\cos t + i \sin t) + B(\cos t - i \sin t) = (A + B)\cos t + i(A - B)\sin t, \tag{35}$$

or

$$u(t) = A_1 \cos t + A_2 \sin t, \tag{36}$$

with (new) arbitrary constants A_1 and A_2. The initial condition $u(0) = \alpha$ yields

$$u(0) = A_1 \cos 0 + A_2 \sin 0 = A_1 = \alpha. \tag{37}$$

For the second condition $u'(0) = \beta$, we need to compute the first derivative

$$u'(t) = -A_1 \sin t + A_2 \cos t, \tag{38}$$

and evaluate it at $t = 0$:

$$u'(0) = -A_1 \sin 0 + A_2 \cos 0 = A_2 = \beta. \tag{39}$$

Thus the solution to the initial value problem

$$u'' + u = 0, \qquad u(0) = \alpha, \qquad u'(0) = \beta, \tag{40}$$

is found to be

$$u(t) = \alpha \cos t + \beta \sin t. \tag{41}$$

Since (37) and (39) give the constants A_1 and A_2 immediately, one might say that the set $\{\cos t, \sin t\}$ is quite well suited for the initial value problem.

Example 20. One choice for the set $\{\phi_1(t), \phi_2(t)\}$ would be $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ if $\lambda_1 \neq \lambda_2$, and $\{e^{\lambda_1 t}, te^{\lambda_2 t}\}$ if $\lambda_1 = \lambda_2$, where λ_1 and λ_2 are the characteristic roots. Assuming $\lambda_1 \neq \lambda_2$ for simplicity, the analogues of (37) and (39) are then

$$u(0) = A_1 e^{\lambda_1 \cdot 0} + A_2 e^{\lambda_2 \cdot 0} = A_1 + A_2 = \alpha, \tag{42}$$

and

$$u'(0) = \lambda_1 A_1 + \lambda_2 A_2 = \beta, \tag{43}$$

the latter because

$$u'(t) = \lambda_1 A_1 e^{\lambda_1 t} + \lambda_2 A_2 e^{\lambda_2 t}. \tag{44}$$

The system (42)-(43) can be solved for A_1 and A_2 as

$$A_1 = \frac{\lambda_2 \alpha - \beta}{\lambda_2 - \lambda_1}, \qquad A_2 = \frac{\beta - \lambda_1 \alpha}{\lambda_2 - \lambda_1}, \tag{45}$$

leading to

$$u(t) = \frac{(\lambda_2 \alpha - \beta) e^{\lambda_1 t} + (\beta - \lambda_1 \alpha) e^{\lambda_2 t}}{\lambda_2 - \lambda_1}. \tag{46}$$

Exercise 21. What equations analogous to (42) and (43) do A_1 and A_2 need to satisfy in case $\phi_1(t) = e^{\lambda_1 t}$ and $\phi_2(t) = te^{\lambda_2 t}$ with $\lambda_1 = \lambda_2$?

It is now clear that in the general case (32), one needs to solve the system

$$\begin{cases} A_1 \phi_1(0) + A_2 \phi_2(0) = \alpha, \\ A_1 \phi_1'(0) + A_2 \phi_2'(0) = \beta, \end{cases} \tag{47}$$

for A_1 and A_2, which is solvable for any given scalars α and β if and only if

$$\begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = \phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0) \neq 0. \tag{48}$$

Example 22. For Example 20, the system (47) obviously reduces to (42) and (43), and the solvability condition (48) becomes

$$\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_2 - \lambda_1 \neq 0, \tag{49}$$

which holds as long as $\lambda_1 \neq \lambda_2$.

Exercise 23. Analyse the case $\phi_1(t) = e^{\lambda_1 t}$ and $\phi_2(t) = te^{\lambda_2 t}$ with $\lambda_1 = \lambda_2$.

Exercise 24. Show that the solutions to (47) are given by

$$A_1 = \frac{\alpha\phi_2'(0) - \beta\phi_2(0)}{W}, \qquad \text{and} \qquad A_2 = \frac{\beta\phi_1(0) - \alpha\phi_1'(0)}{W}, \tag{50}$$

where $W = \phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)$.

Remark 25. Since ϕ_1 and ϕ_2 are solutions of the equation $u''+pu'+qu=0$, and (47) ensures that the initial conditions are correct, the resulting function $u = A_1\phi_1 + A_2\phi_2$ satisfies the initial value problem (31).

Of course, there is nothing special about the point $t = 0$, and we could also consider an initial value problem where we fix the initial conditions at some specified point $t = s$, as

$$\begin{cases} u'' + pu' + qu = 0, \\ u(s) = \alpha, \\ u'(s) = \beta. \end{cases} \tag{51}$$

In this case, we will need the following determinant to be nonzero:

$$W(s) = W[\phi_1, \phi_2](s) = \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1'(s) & \phi_2'(s) \end{vmatrix} = \phi_1(s)\phi_2'(s) - \phi_1'(s)\phi_2(s). \tag{52}$$

Definition 26. The quantity $W(s)$ is called the *Wronskian* of ϕ_1 and ϕ_2 at s.

Thus, given two solutions ϕ_1 and ϕ_2 of the equation $u''+pu'+qu=0$ with $W[\phi_1, \phi_2](s) \neq 0$, we would solve the linear system

$$\begin{cases} A_1\phi_1(s) + A_2\phi_2(s) = \alpha, \\ A_1\phi_1'(s) + A_2\phi_2'(s) = \beta, \end{cases} \tag{53}$$

for A_1 and A_2. Then $u = A_1\phi_1 + A_2\phi_2$ is a solution to the initial value problem (51).

Exercise 27. Develop the answer in (50) a bit further to get the expression

$$u(t) = \frac{\alpha}{W(s)} \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(s) & \phi_2'(s) \end{vmatrix} + \frac{\beta}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix}, \tag{54}$$

for the aforementioned solution of (51).

7. Abel's formula

Let us compute the derivative of the Wronskian:

$$W' = (\phi_1\phi_2' - \phi_1'\phi_2)' = \phi_1'\phi_2' + \phi_1\phi_2'' - \phi_1''\phi_2 - \phi_1'\phi_2' = \phi_1\phi_2'' - \phi_1''\phi_2, \tag{55}$$

and take into account the fact that both ϕ_1 and ϕ_2 satisfy $u'' + pu' + qu = 0$:

$$W' = \phi_1(-p\phi_2' - q\phi_2) - (-p\phi_1' - q\phi_1)\phi_2 = -p(\phi_1\phi_2' - \phi_1'\phi_2) = -pW. \tag{56}$$

In other words, we have

$$W(s) = W(0)e^{-ps} \qquad \text{for} \quad s \in \mathbb{R}, \tag{57}$$

or

$$W(s) = W(t)e^{-p(s-t)} \qquad \text{for any} \quad s, t \in \mathbb{R}. \tag{58}$$

Any of the last 3 formulas may be called *Abel's formula*. This immediately implies the following theorem, sometimes called Abel's theorem.

Theorem 28. *Let ϕ_1 and ϕ_2 be solutions of the differential equation $u''+pu'+qu=0$. Then their Wronskian is either identically zero, or nonzero everywhere.*

Thus for solutions to differential equations of the type under consideration, we can talk about their Wronskian being simply zero or nonzero, without specifying where we evaluate the Wronskian.

8. Fundamental sets

In Section 6, we have constructed a solution to the initial value problem, by utilizing two given solutions ϕ_1 and ϕ_2 of the differential equation. We saw that the Wronskian $W[\phi_1, \phi_2]$ needs to be nonzero. A natural question that arises is if the solution given by this procedure is the only possible solution to the initial value problem. In other words, we consider now the question of uniqueness of solutions for the initial value problem. At this point, it will be convenient to introduce the following terminology.

Definition 29. A *fundamental set* (or *a basis*) of solutions of the equation $u'' + pu' + qu = 0$ is a set $\{\phi_1, \phi_2\}$ of solutions of the same equation, satisfying $W[\phi_1, \phi_2] \neq 0$.

More generally, for any two functions v and w, if $W[v, w](s) \neq 0$ then we say that v and w are *linearly independent at* s.

Example 30. The sets $\{\cos t, \sin t\}$ and $\{e^{it}, e^{-it}\}$ both are fundamental sets of solutions of the differential equation $u'' + u = 0$.

The following is a slight generalization of Theorem 12. This gives an affirmative answer to the uniqueness question, because it says that the solutions of the form $u = A_1\phi_1 + A_2\phi_2$ cover the entire solution space, and our construction in Section 6 picks the unique one among the solutions of the that form.

Theorem 31. *Let $\{\phi_1, \phi_2\}$ be a fundamental set of solutions of the equation $u'' + pu' + qu = 0$. Then for any solution u of the same equation, there exists a unique pair of constants A_1 and A_2, such that $u = A_1\phi_1 + A_2\phi_2$.*

Proof. Let $v = A_1\phi_1 + A_2\phi_2$ be the solution of the initial value problem with the initial data given by those of u, that is, let v be constructed as in Section 6, satisfying

$$\begin{cases} v'' + pv' + qv = 0, \\ v(0) = u(0), \\ v'(0) = u'(0). \end{cases} \tag{59}$$

Since v satisfies the same initial conditions as u, the representation formula (26) guarantees that $v(t) = u(t)$ for all t, meaning that u itself is of the form $A_1\phi_1 + A_2\phi_2$. □

In view of (54), the following is immediate.

Corollary 32. *Any solution of $u'' + pu' + qu = 0$ in any interval (a, b) has the representation*

$$u(t) = \frac{u(s)}{W(s)} \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(s) & \phi_2'(s) \end{vmatrix} + \frac{u'(s)}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix}, \qquad t \in (a, b), \tag{60}$$

where $s \in (a, b)$ and $\{\phi_1, \phi_2\}$ is a fundamental set of solutions of $u'' + pu' + qu = 0$.

9. Inhomogenous equations

We are now ready to look at the inhomogenous equation

$$u'' + pu' + qu = f, \tag{61}$$

where $f = f(t)$ is a given function, called the *forcing function*, the *source term*, etc. The solution space of (61) is an *affine space*, in the sense that the difference between any two solutions of (61) satisfies the corresponding homogeneous equation.

Exercise 33. Show that if u and v are solutions of (61), then the difference $w = u - v$ satisfies

$$w'' + pw' + qw = 0. \tag{62}$$

Let us fix a function v satisfying (61). Then the preceding exercise implies that any other solution u of (61) is of the form $u = v + w$, where w is some solution of (62).

Exercise 34. Show that if v satsfies (61), and w satisfies (62), then $u = v + w$ satisfies (61).

Remark 35. Thus the general solution of (61) can be constructed as follows.

- Find a single *particular solution* v of the inhomogeneous equation (61).
- Find the general solution w of the *complementary equation* (62). In particular, this solution will contain 2 arbitrary constants.
- Then the sum $u = v + w$ is the general solution of (61).

Since we already know how to find the general solution of the homogenous equation (62), the focus now is on finding a single particular solution of (61).

Example 36. Consider the equation

$$u'' + 2u' + u = e^{3t}. \tag{63}$$

The characteristic roots are $\lambda_{1,2} = -1$, and so the general solution of the complementary equation is

$$w(t) = Ae^{-t} + Bte^{-t}. \tag{64}$$

We look for a particular solution to (63) in the form

$$v(t) = Ce^{3t}, \tag{65}$$

where C is a parameter that should be adjusted. Computing the derivatives of v and substituting into (63), we get

$$9Ce^{3t} + 2 \cdot 3Ce^{3t} + Ce^{3t} = e^{3t}, \tag{66}$$

giving $C = 1/16$. Finally, we conclude that the general solution of (63) is

$$u(t) = \frac{e^{3t}}{16} + Ae^{-t} + Bte^{-t}, \tag{67}$$

where A and B are arbitrary constants.

As this and the following examples show, in some cases one can simply make a guess at the form of a particular solution.

Example 37. Consider the equation

$$u'' - u = t^2. \tag{68}$$

Looking for a particular solution in the form $v(t) = At^2 + B$ yields $v(t) = -t^2 - 2$.

Example 38. Consider the equation

$$x'' + x' + x = te^{-t}. \tag{69}$$

Looking for a particular solution in the form $y(t) = Ae^{-t} + Bte^{-t}$ yields $A = B = 1$.

The following exercise shows that if the forcing function is the sum of several terms, then each of those terms can be treated separately.

Exercise 39. Show that if $v_1'' + pv_1' + qv_1 = f_1$ and $v_2'' + pv_2' + qv_2 = f_2$, then the sum $v = v_1 + v_2$ satisfies $v'' + pv' + qv = f$ with $f = f_1 + f_2$.

Exercise 40. Find the general solution of $x'' + x = t^2 + \sin t$.

10. Variation of constants

In this section, we will learn a way to construct a solution of the inhomogeneous equation

$$u'' + pu' + qu = f. \tag{70}$$

Let $\{\phi_1, \phi_2\}$ be a fundamental set of solutions to the complementary equation

$$w'' + pw' + qw = 0. \tag{71}$$

Since the Wronskian $W = \phi_1\phi_2' - \phi_1'\phi_2$ is nonzero everywhere, ϕ_1 and ϕ_2 cannot vanish at the same time, and therefore the solution u can be written as

$$u(t) = A_1(t)\phi_1(t) + A_2(t)\phi_2(t), \tag{72}$$

with some coefficients $A_1(t)$ and $A_2(t)$ depending on t. This is called the method of *variation of constants* (or of *variation of parameters*), since if A_1 and A_2 are constants then (72) solves the homogeneous equation (71).

Assuming that A_1 and A_2 can be differentiated twice, we compute

$$\begin{aligned} u' &= A_1'\phi_1 + A_1\phi_1' + A_2'\phi_2 + A_2\phi_2', \\ u'' &= A_1''\phi_1 + 2A_1'\phi_1' + A_1\phi_1'' + A_2''\phi_2 + 2A_2'\phi_2' + A_2\phi_2''. \end{aligned} \tag{73}$$

Since

$$\begin{aligned} A_1''\phi_1 + 2A_1'\phi_1' + A_1\phi_1'' + pA_1'\phi_1 + pA_1\phi_1' + qA_1\phi_1 &= A_1''\phi_1 + 2A_1'\phi_1' + pA_1'\phi_1 \\ &= (A_1'\phi_1)' + A_1'\phi_1' + pA_1'\phi_1. \end{aligned} \tag{74}$$

we have

$$u'' + pu' + qu = (A_1'\phi_1)' + pA_1'\phi_1 + A_1'\phi_1' + (A_2'\phi_2)' + pA_2'\phi_2 + A_2'\phi_2'. \tag{75}$$

This means that if

$$\begin{aligned} A_1'\phi_1 + A_2'\phi_2 &= 0, \\ A_1'\phi_1' + A_2'\phi_2' &= f, \end{aligned} \tag{76}$$

then we would have

$$u'' + pu' + qu = f. \tag{77}$$

The system (76) can be solved for A_1' and A_2', as

$$A_1'(t) = -\frac{f(t)\phi_2(t)}{W(t)} \qquad \text{and} \qquad A_2'(t) = \frac{f(t)\phi_1(t)}{W(t)}, \tag{78}$$

where $W = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$. We make this argument precise in the following theorem.

Theorem 41. *Let $f \in C[a,b)$ with $-\infty < a < b \leq \infty$, and let u be a function twice differentiable in (a,b), with $u' \in C[a,b)$ and*

$$u'' + pu' + qu = f \quad \text{in } (a,b). \tag{79}$$

Then u is given by

$$u(t) = w(t) + \int_a^t \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{W(s)} f(s)ds, \qquad t \in [a,b), \tag{80}$$

where w is the solution of the homogeneous initial value problem

$$\begin{cases} w'' + pw' + qw = 0, \\ w(a) = u(a), \\ w'(a) = u'(a). \end{cases} \tag{81}$$

Proof. For $t \in [a,b)$, let

$$A_1(t) = -\int_a^t \frac{f(s)\phi_2(s)}{W(s)} ds \qquad \text{and} \qquad A_2(t) = \int_a^t \frac{f(s)\phi_1(s)}{W(s)} ds. \tag{82}$$

Then according to the fundamental theorem of calculus, A_1 and A_2 are differentiable in (a,b), and $A_1, A_2 \in C[a,b)$, with

$$A_1' = -\frac{f\phi_2}{W} \quad \text{and} \quad A_2' = \frac{f\phi_1}{W} \qquad \text{in} \quad (a,b). \tag{83}$$

This implies that

$$\begin{aligned} A_1'\phi_1 + A_2'\phi_2 &= 0, \\ A_1'\phi_1' + A_2'\phi_2' &= f, \end{aligned} \tag{84}$$

in (a,b). Now let

$$v(t) = A_1(t)\phi_1(t) + A_2(t)\phi_2(t) = \int_0^t \frac{-\phi_2(s)\phi_1(t) + \phi_1(s)\phi_2(t)}{W(s)} f(s) ds, \tag{85}$$

for $t \in [a,b)$, meaning that $v \in C[a,b)$ and that v is differentiable in (a,b) with

$$v' = A_1'\phi_1 + A_1\phi_1' + A_2'\phi_2 + A_2\phi_2' = A_1\phi_1' + A_2\phi_2' \qquad \text{in} \quad (a,b). \tag{86}$$

Since $A_1, A_2 \in C[a,b)$, we have $v' \in C[a,b)$. Moreover, note that $v(a) = v'(a) = 0$.

Next, we compute

$$v'' = A_1'\phi_1' + A_1\phi_1'' + A_2'\phi_2' + A_2\phi_2'' = f + A_1\phi_1'' + A_2\phi_2'' \qquad \text{in} \quad (a,b), \tag{87}$$

which implies that

$$v'' + pv' + qv = f \qquad \text{in} \quad (a,b). \tag{88}$$

Hence the difference $w = u - v$ satisfies $w'' + pw' + qw = 0$ in (a,b). We also have $w' \in C[a,b)$ with $w(a) = u(a)$ and $w'(a) = u'(a)$. By the theory of homogeneous equations, there is exactly one such function. The proof is complete. □

Corollary 42. *Let $f \in C[a,b)$ with $-\infty < a < b \leq \infty$, and let u be a function twice differentiable in (a,b), with $u' \in C[a,b)$ and $u'' + pu' + qu = f$ in (a,b). Then u has the representation*

$$u(t) = \frac{u(a)}{W(a)} \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(a) & \phi_2'(a) \end{vmatrix} + \frac{u'(a)}{W(a)} \begin{vmatrix} \phi_1(a) & \phi_2(a) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} + \int_a^t \frac{f(s)}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} ds, \tag{89}$$

where $\{\phi_1, \phi_2\}$ is a fundamental set of solutions of $w'' + pw' + qw = 0$.

Exercise 43. Prove the preceding corollary.

11. Forced harmonic oscillators

Consider the equation

$$x'' + x = e^{i\omega t}. \tag{90}$$

Looking for a particular solution in the form $y(t) = ae^{i\omega t}$ yields $a = 1/(1-\omega^2)$, provided that $\omega \neq 1$. Combining this particular solution with the general solution of the complementary equation, we get

$$x(t) = \frac{e^{i\omega t}}{1-\omega^2} + Ae^{it} + Be^{-it} \qquad (\omega \neq 1). \tag{91}$$

In case $\omega = 1$ we use the form $y(t) = ate^{i\omega t}$, which gives

$$x(t) = \frac{te^{it}}{2i} + Ae^{it} + Be^{-it}. \tag{92}$$

Exercise 44. Show that the equation

$$x'' + \omega_0^2 x = e^{i\omega t} \qquad (\omega_0 > 0), \tag{93}$$

can be written in the form (90) by suitably scaling the t-variable.

Now we look at the equation

$$x'' + 2\mu x' + x = e^{i\omega t}, \tag{94}$$

with $\mu > 0$. Assuming zero initial data for simplicity, a direct application of the variation of parameters formula (80) yields

$$x(t) = \frac{e^{i\omega t}}{(i\omega - \lambda_1)(i\omega - \lambda_2)} + \frac{e^{\lambda_2 t}}{2\Delta(i\omega - \lambda_1)} - \frac{e^{\lambda_1 t}}{2\Delta(i\omega - \lambda_2)}, \tag{95}$$

where $\lambda_{1,2}$ are the characteristic roots and $\Delta = \sqrt{\mu^2 - 1}$. Since $\mathrm{Re}(\lambda_{1,2}) < 0$, the second and the third terms tend to 0 as $t \to \infty$. These terms are called *transient*. When t is large, what remains is the *steady state*, or the *response* term

$$x(t) = \frac{e^{i\omega t}}{(i\omega - \lambda_1)(i\omega - \lambda_2)}. \tag{96}$$

Note that the denominator $(i\omega - \lambda_1)(i\omega - \lambda_2)$ is simply the characteristic polynomial of (94) evaluated at $i\omega$.

The preceding observation holds generally true: Given any equation

$$x'' + px' + qx = e^{i\omega t}, \tag{97}$$

with the characteristic polynomial

$$P(\lambda) = \lambda^2 + p\lambda + q, \tag{98}$$

looking for a particular solution in the form $x(t) = Ce^{i\omega t}$ gives

$$x(t) = \frac{e^{i\omega t}}{P(i\omega)}. \tag{99}$$

This solution will be the steady state if the complementary equation (i.e., the "unforced" system) is strictly stable, as any realistic system would be.

Definition 45. The *transfer function* of the system is

$$T(\omega) = \frac{1}{P(i\omega)}. \tag{100}$$

Moreover, writing $T(\omega) = \rho e^{-i\gamma}$, the magnitude ρ is called the *gain*, and the negative argument γ is called the *phase lag* of the system.

These terms are obviously inspired by the fact that the input forcing $e^{i\omega t}$ is "transferred" to the output solution

$$x(t) = T(\omega)e^{i\omega t} = \rho e^{i(\omega t - \gamma)}. \tag{101}$$

Example 46. Suppose that the system

$$x'' + px' + qx = e^{i\omega_1 t} + 2e^{i\omega_2 t}, \tag{102}$$

has the transfer function T, and the corresponding complementary equation is strictly stable. Then the steady state is given by

$$x(t) = T(\omega_1)e^{i\omega_1 t} + 2T(\omega_2)e^{i\omega_2 t}. \tag{103}$$

Example 47. The gain function for the system (94) is

$$|T(\omega)| = \frac{1}{\sqrt{(\omega^2 - 1)^2 + 4\mu^2\omega^2}}. \tag{104}$$

12. Regularity

In Section 3 we proved that any solution of

$$u' + pu = f, \tag{105}$$

has the representation formula

$$u(t) = w(t) + \int_a^t e^{-p(t-s)} f(s) ds, \qquad t \in [a, b), \tag{106}$$

where w is the solution of the complementary equation $w' + pw = 0$ satisfying $w(a) = u(a)$. Here $-\infty < a < b \leq \infty$, and we have assumed that $u, f \in C[a, b)$ with u differentiable in the interval (a, b).

Now observe that we can find u' from (105) and write

$$u' = f - pu, \tag{107}$$

which, on account of the conditions $u, f \in C[a, b)$ shows that u' is indeed continuous in $[a, b)$. This fact is written as $u \in C^1[a, b)$.

Furthermore, if f is *differentiable* in (a, b), then we can differentiate (107), and get

$$u'' = f' - pu', \tag{108}$$

meaning that u is twice differentiable in (a, b). This formula also shows that if $f \in C^1(a, b)$ then $u \in C^2(a, b)$ and if $f \in C^1[a, b)$ then $u \in C^2[a, b)$. Here, $u \in C^2(a, b)$ means that the second derivative of u exists in (a, b), and $u \in C^2[a, b)$ means in addition that u'' can be extended to a continuous function in $[a, b)$.

Repeating the argument, we infer the following.

- If f is k times differentiable in (a, b), then u is $k + 1$ times differentiable in (a, b).
- If $f \in C^k(a, b)$ then $u \in C^{k+1}(a, b)$.
- If $f \in C^k[a, b)$ then $u \in C^{k+1}[a, b)$.

We may say that the solution of (105) is *as regular as the right hand side allows it to be.*

Similarly to the first order case, in Section 10 we proved that any solution of

$$u'' + pu' + qu = f, \tag{109}$$

has the representation formula

$$u(t) = w(t) + \int_a^t \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{\phi_2'(s)\phi_1(s) - \phi_1'(s)\phi_2(s)} f(s) ds, \qquad t \in [a, b), \tag{110}$$

where w is the solution of $w'' + pw' + qw = 0$ satisfying $w(a) = u(a)$ and $w'(a) = u'(a)$. Here $-\infty < a < b \leq \infty$, and we have assumed that $\{\phi_1, \phi_2\}$ is a fundamental set of solutions, and that $u', f \in C[a, b)$ with u twice differentiable in (a, b).

Imitating what we did in the first order case, we write

$$u'' = f - pu' - qu, \tag{111}$$

and infer that in fact $u \in C^2[a, b)$. Moreover, by considering additional regularity assumptions on f, we get the following.

- If f is k times differentiable in (a, b), then u is $k + 2$ times differentiable in (a, b).
- If $f \in C^k(a, b)$ then $u \in C^{k+2}(a, b)$.
- If $f \in C^k[a, b)$ then $u \in C^{k+2}[a, b)$.

Thus the solution of (109) is *as regular as the right hand side allows it to be.*

Having studied what happens when the source term f has additinoal smoothness, a natural question would be to ask: What if f is *not* continuous? In this case, at the points of discontinuities of f, the function u given by (106) (resp. (110)) would *not* satisfy the corresponsing equation (105) (resp. (109)). We say that u is *not a classical solution.* However, especially if

these exceptional points occupy a negligible portion of the interval $(0, T)$, it is reasonable to accept u as a solution in some generalized, weaker sense.

Definition 48. The function u given by (106) (respectively (110)) is called a *weak solution* of the corresponsing equation (105) (respectively (109)), whenever the integral appearing in the formula (106) (respectively (110)) makes sense.

To make this definition completely precise, we need to agree on how we interpret the said integrals. The most common options are the Riemann integral, improper Riemann integrals, and the Lebesgue integral.

Remark 49 (Piecewise continuous forcing)**.** The simplest instance of such a generalization occurs when f has a finitely many discontinuities, and each if its continuous pieces is Riemann integrable (or at least improperly integrable). In this case, u from (106) is *merely continuous* across the discontinuities of f. The equation (105) loses its meaning only at those points.

On the other hand, u from (110) has the property that u' is also continuous across the discontinuities of f. This can be seen from the proof of Theorem 41, especially the formula (86). Again, the equation (109) loses its meaning only at those points.

The philosophy here is that by ignoring a few points at the level of the differential equation, we are not losing any essential information about the underlying system.

Example 50. Let us find the general weak solution of

$$x'' + x = \theta, \tag{112}$$

where θ is the Heaviside step function

$$\theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{113}$$

We intentionally left the value $\theta(0)$ *undefined*, because anything of importance should not depend on this value. Instead of using the formula (110) we will solve (112) piece by piece, and then glue the pieces together. The weak solution of (110) should be equal to a solution of the homogeneous equation $x'' + x = 0$ for $t < 0$, and equal to a solution of $x'' + x = 1$ for $t > 0$. Thus we can write

$$x(t) = \begin{cases} A\cos t + B\sin t, & t < 0, \\ C\cos t + D\sin t + 1, & t > 0. \end{cases} \tag{114}$$

Should the general solution of (112) really involve 4 arbitrary constants? No, the constants A, B, C, and D will have codependence because $x(t)$ and $x'(t)$ should be continuous across the point $t = 0$. Imposing these continuity conditions yield $A = C + 1$ and $B = D$, meaning that the general solution of (112) is

$$x(t) = A\cos t + B\sin t + \theta(t)(1 - \cos t), \tag{115}$$

where A and B are arbitrary constants.

Remark 51 (Measurable functions)**.** A far reaching generalization of the piecewise continuous case is obtained by allowing f to be Lebesgue integrable in each closed subinterval of $[0, T)$, and interpreting the integrals in the representation formulae as the Lebesgue integral. Then as a consequence of the Lebesgue differentiation theorem, the corresponding differential equations are satisfied almost everywhere in $(0, T)$. Specifically, the equations are satisfied wherever f is continuous. Moreover, u from (106) is continuous in $[0, T)$, and both u and its derivative from (110) are continuous in $[0, T)$.

Remark 52 (Delta measure)**.** We can even go beyond functions by allowing measures such as the Dirac delta. In this case, even the (weak) solution u can become discontinuous.

Roughly speaking, a measure is something that can be integrated. They do not in general have pointwise values. For example, the Dirac delta (at $z \in \mathbb{R}$) is defined by the property

$$\int_a^b \delta_z(s) ds = \begin{cases} 1, & z \in [a,b], \\ 0, & z \notin [a,b]. \end{cases} \tag{116}$$

We set $\delta = \delta_0$. Another important (and in fact defining) property is

$$\int_a^b \delta_z(s) f(s) ds = \begin{cases} f(z), & z \in [a,b], \\ 0, & z \notin [a,b]. \end{cases} \tag{117}$$

for any continuous function f.

Example 53. Let us find the general weak solution of

$$x' + x = \delta. \tag{118}$$

By (106), we have

$$x(t) = y(t) + \int_{-1}^t e^{-(t-s)} \delta(s) ds = y(t) + \theta(t) e^{-t}, \tag{119}$$

where y is the general solution of $y' + y = 0$.

Exercise 54. Show that (106) with $f = \delta_z$ becomes

$$u(t) = w(t) + \theta(t-z) e^{-p(t-z)}. \tag{120}$$

Example 55. Let us find the general weak solution of

$$x'' + x = \delta_\pi. \tag{121}$$

Take the fundamental set $\{\cos t, \sin t\}$, and apply (110), to get

$$x(t) = y(t) + \int_0^t (\sin t \cos s - \cos t \sin s) \delta_\pi(s) ds = y(t) - \theta(t-\pi) \sin t, \tag{122}$$

where y is the general solution of $y'' + y = 0$. Interestingly, $x(t)$ is continuous at $t = \pi$, even though $x'(t)$ has a discontinuity there.

Exercise 56. Show that (110) with $f = \delta_z$ becomes

$$u(t) = w(t) + \theta(t-z) \frac{\phi_2(t)\phi_1(z) - \phi_1(t)\phi_2(z)}{\phi_2'(z)\phi_1(z) - \phi_1'(z)\phi_2(z)}, \tag{123}$$

and show that this u is continuous at $t = z$.

13. Higher order homogeneous equations

The most general form of a constant coefficient linear homogeneous equation is

$$u^{(n)} + a_{n-1} u^{(n-1)} + \ldots + a_1 u' + a_0 u = 0, \tag{124}$$

where $a_0, \ldots, a_{n-1}$ are given constants, and $u = u(t)$ is of course the unknown solution. Looking for the solution in the form $u(t) = e^{\lambda t}$ leads to the *characteristic equation*

$$p(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0. \tag{125}$$

We call p the *characteristic polynomial*, and its roots the *characteristic roots*. With the help of the characteristic polynomial, we can write (124) in a convenient form as

$$p(D) u = 0, \tag{126}$$

where $D = d/dt$, and the expression $p(D)u$ should be understood as

$$\begin{aligned} p(D)u &= (D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)u \\ &= D^n u + a_{n-1}D^{n-1}u + \ldots + a_1 Du + a_0 u. \end{aligned} \tag{127}$$

Since $D^k e^{\lambda t} = \lambda^k e^{\lambda t}$, it is easy to see that

$$p(D)e^{\lambda_j t} = p(\lambda_j)e^{\lambda_j t} = 0, \tag{128}$$

for any characterisic root λ_j, meaning that $u(t) = e^{\lambda_j t}$ is a solution to (124).

Lemma 57. *Suppose that λ_j occurs with multiplicity m as a root of p. Then the functions*

$$e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{m-1}e^{\lambda_j t}, \tag{129}$$

are all solutions to (124).

Proof. We can write

$$p(\lambda) = (\lambda - \lambda_j)^m q(\lambda), \tag{130}$$

with a polynomial q with $q(\lambda_j) \neq 0$, and so

$$p(\lambda_j) = p'(\lambda_j) = \ldots = p^{(m-1)}(\lambda_j) = 0. \tag{131}$$

Since $p(D)e^{\lambda t} = p(\lambda)e^{\lambda t}$, we get

$$\begin{aligned} \frac{d}{d\lambda}\big(p(D)e^{\lambda t}\big) &= \frac{d}{d\lambda}\big(p(\lambda)e^{\lambda t}\big) = p'(\lambda)e^{\lambda t} + p(\lambda)te^{\lambda t}, \\ \frac{d^2}{d\lambda^2}\big(p(D)e^{\lambda t}\big) &= \frac{d^2}{d\lambda^2}\big(p(\lambda)e^{\lambda t}\big) = p''(\lambda)e^{\lambda t} + 2p'(\lambda)te^{\lambda t} + p(\lambda)t^2 e^{\lambda t}, \\ &\ldots \\ \frac{d^k}{d\lambda^k}\big(p(D)e^{\lambda t}\big) &= p^{(k)}(\lambda)e^{\lambda t} + kp^{k-1}(\lambda)te^{\lambda t} + \ldots + p(\lambda)t^k e^{\lambda t}, \end{aligned} \tag{132}$$

implying that

$$\frac{d^k}{d\lambda^k}\big(p(D)e^{\lambda t}\big)\Big|_{\lambda=\lambda_j} = 0, \qquad \text{for} \quad k = 0, \ldots, m-1. \tag{133}$$

On the other hand, we have

$$\frac{d^k}{d\lambda^k}\big(p(D)e^{\lambda t}\big) = p(D)\frac{d^k e^{\lambda t}}{d\lambda^k} = p(D)(t^k e^{\lambda t}), \tag{134}$$

which gives

$$p(D)(t^k e^{\lambda_j t}) = 0, \qquad \text{for} \quad k = 0, \ldots, m-1. \tag{135}$$

The proof is complete. □

Thus if $\lambda_1, \ldots, \lambda_n$ are the characteristic roots, counted according to their multiplicities, then for any values of the constants $A_1, \ldots, A_n$, the function

$$u(t) = A_1 t^{k_1} e^{\lambda_1 t} + A_2 t^{k_2} e^{\lambda_2 t} + \ldots + A_n t^{k_n} e^{\lambda_n t}, \tag{136}$$

solves (124), where k_j counts how many times the value λ_j appears in the list $\lambda_1, \ldots, \lambda_{j-1}$.

Example 58. The characteristic roots for the equation

$$u''' - 3u' + 2u = 0, \tag{137}$$

are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = -2$, and so

$$u(t) = A_1 e^t + A_2 te^t + A_3 e^{-2t}, \tag{138}$$

is a solution. Note that $k_1 = 0$, $k_2 = 1$, and $k_2 = 0$.

As expected, it turns out that (136) is the general solution of (124).

Definition 59. A *classical solution* (or simply a *solution*) of (124) is a function that is n times differentiable everywhere and satisfies (124) everywhere in $\mathbb{R}$.

Theorem 60. *Any classical solution u of (124) is of the form (136). In fact, we have the representation formula*

$$\begin{aligned} u(t) &= C_0 e^{\lambda_1 t} + C_1 e^{\lambda_1 t} \int_0^t e^{(\lambda_2-\lambda_1)s} ds + C_2 e^{\lambda_1 t} \int_0^t e^{(\lambda_2-\lambda_1)s_1} \int_0^{s_1} e^{(\lambda_3-\lambda_2)s_2} ds_2 ds_1 \\ &\quad + \dots \\ &\quad + C_{n-1} e^{\lambda_1 t} \int_0^t e^{(\lambda_2-\lambda_1)s_1} \int_0^{s_1} e^{(\lambda_3-\lambda_2)s_2} \dots \int_0^{s_{n-2}} e^{(\lambda_n-\lambda_{n-1})s_{n-1}} ds_{n-1} \cdots ds_1, \end{aligned} \tag{139}$$

where $C_0 = u(0)$ and

$$C_k = \big[(D-\lambda_1)\cdots(D-\lambda_k)u\big]\Big|_{t=0} \qquad (k = 1, \dots, n-1). \tag{140}$$

Proof. Let u be a classical solution of (16), and let $\lambda_1, \dots, \lambda_n$ be the characteristic roots. Then we define the functions

$$\begin{aligned} v_1 &= u' - \lambda_1 u = (D-\lambda_1)u, \\ v_2 &= v_1' - \lambda_2 v_1 = (D-\lambda_2)(D-\lambda_1)u, \\ &\dots \\ v_{n-1} &= v_{n-2}' - \lambda_{n-1} v_{n-2} = (D-\lambda_{n-1})\cdots(D-\lambda_1)u \end{aligned} \tag{141}$$

and note that

$$v_{n-1}' - \lambda_n v_{n-1} = (D-\lambda_n)v_{n-1} = (D-\lambda_n)\cdots(D-\lambda_1)u = p(D)u = 0. \tag{142}$$

This means that

$$\begin{aligned} v_{n-1}(t) &= v_{n-1}(0) e^{\lambda_n t}, \\ v_{n-2}(t) &= v_{n-2}(0) e^{\lambda_{n-1} t} + v_{n-1}(0) e^{\lambda_{n-2} t} \int_0^t e^{(\lambda_n - \lambda_{n-1})s} ds, \end{aligned} \tag{143}$$

et cetera, we get the formula (139). It is easy to see that (139) is of the form (136). □

Exercise 61. Adapt the preceding theorem to the case of (a, b) instead of $\mathbb{R}$.

14. Fundamental sets

Suppose that the general solution of $p(D)u = 0$ is given by

$$u(t) = A_1 \phi_1(t) + \dots + A_n \phi_n(t), \tag{144}$$

where $\phi_1, \dots, \phi_n$ are some particular solutions of $p(D)u = 0$. Then a way to solve the initial value problem

$$\begin{cases} p(D)u = 0, \\ u(s) = \alpha_0, \\ u'(s) = \alpha_1, \\ \dots \\ u^{(n-1)}(s) = \alpha_{n-1}, \end{cases} \tag{145}$$

would be to solve the linear system

$$\begin{aligned} A_1\phi_1(s) + \ldots + A_n\phi_n(s) &= \alpha_0, \\ A_1\phi_1'(s) + \ldots + A_n\phi_n'(s) &= \alpha_1, \\ &\ldots \\ A_1\phi_1^{(n-1)}(s) + \ldots + A_n\phi_n^{(n-1)}(s) &= \alpha_{n-1}. \end{aligned} \tag{146}$$

For this to be solvable, the determinant

$$W(s) = \begin{vmatrix} \phi_1(s) & \ldots & \phi_n(s) \\ \phi_1'(s) & \ldots & \phi_n'(s) \\ \vdots & \ldots & \vdots \\ \phi_1^{(n-1)}(s) & \ldots & \phi_n^{(n-1)}(s) \end{vmatrix}, \tag{147}$$

needs to be nonzero. As expected, this determinant $W[\phi_1, \ldots, \phi_n](s) = W(s)$ is called the *Wronskian* of $\phi_1, \ldots, \phi_n$ at s. We shall now address the issue of uniqueness.

Definition 62. A *fundamental set* (or *a basis*) of solutions of the equation $p(D)u = 0$ is a set $\{\phi_1, \ldots, \phi_n\}$ of solutions of the same equation, satisfying $W[\phi_1, \ldots, \phi_n](0) \neq 0$.

More generally, for any collection of functions $v_1, \ldots, v_n$, if $W[v_1, \ldots, v_n](s) \neq 0$ then we say that the set $\{v_1, \ldots, v_n\}$ is *linearly independent at* s.

Example 63. The sets $\{\cos t, \sin t, e^t\}$ and $\{e^{it}, e^{-it}, e^t\}$ both are fundamental sets of solutions of the differential equation $u''' - u'' + u' - u = 0$.

The following is a slight generalization of Theorem 60, which gives an affirmative answer to the uniqueness question.

Theorem 64. *Let $\{\phi_1, \ldots, \phi_n\}$ be a fundamental set of solutions of the equation $p(D)u = 0$. Then for any solution u of the same equation, there exists a unique set of constants $A_1, \ldots, A_n$, such that $u = A_1\phi_1 + \ldots + A_n\phi_n$.*

Proof. Let $v = A_1\phi_1 + \ldots + A_n\phi_n$ be the solution of the initial value problem with the initial data given by those of u, that is, let v be constructed as in this section, satisfying

$$\begin{cases} p(D)v = 0, \\ v(s) = u(s), \\ v'(s) = u'(s), \\ \ldots \\ v^{(n-1)}(s) = u^{(n-1)}(s). \end{cases} \tag{148}$$

Since v satisfies the same initial conditions as u, the representation formula (139) guarantees that $v(t) = u(t)$ for all t, meaning that u itself is of the form $A_1\phi_1 + \ldots + A_n\phi_n$. □

Corollary 65. *Any solution of $p(D)u = 0$ in any interval (a, b) has the representation*

$$\begin{aligned} u(t) = \frac{u(s)}{W(s)}\begin{vmatrix} \phi_1(t) & \dots & \phi_n(t) \\ \phi_1'(s) & \dots & \phi_n'(s) \\ \dots & \dots & \dots \\ \phi_1^{(n-2)}(s) & \dots & \phi_n^{(n-2)}(s) \\ \phi_1^{(n-1)}(s) & \dots & \phi_n^{(n-1)}(s) \end{vmatrix} + \frac{u'(s)}{W(s)}\begin{vmatrix} \phi_1(s) & \dots & \phi_n(s) \\ \phi_1(t) & \dots & \phi_n(t) \\ \dots & \dots & \dots \\ \phi_1^{(n-2)}(s) & \dots & \phi_n^{(n-2)}(s) \\ \phi_1^{(n-1)}(s) & \dots & \phi_n^{(n-1)}(s) \end{vmatrix} + \dots \\ + \frac{u^{(n-2)}(s)}{W(s)}\begin{vmatrix} \phi_1(s) & \dots & \phi_n(s) \\ \phi_1'(s) & \dots & \phi_n'(s) \\ \dots & \dots & \dots \\ \phi_1(t) & \dots & \phi_n(t) \\ \phi_1^{(n-1)}(s) & \dots & \phi_n^{(n-1)}(s) \end{vmatrix} + \frac{u^{(n-1)}(s)}{W(s)}\begin{vmatrix} \phi_1(s) & \dots & \phi_n(s) \\ \phi_1'(s) & \dots & \phi_n'(s) \\ \dots & \dots & \dots \\ \phi_1^{(n-2)}(s) & \dots & \phi_n^{(n-2)}(s) \\ \phi_1(t) & \dots & \phi_n(t) \end{vmatrix} \end{aligned} \tag{149}$$

for $t \in (a, b)$, where $s \in (a, b)$ and $\{\phi_1, \dots, \phi_n\}$ is a fundamental set of solutions of $p(D)u = 0$.

Proof. Let us rewrite (146) here as

$$\begin{bmatrix} \phi_1(s) & \dots & \phi_n(s) \\ \phi_1'(s) & \dots & \phi_n'(s) \\ \dots & \dots & \dots \\ \phi_1^{(n-1)}(s) & \dots & \phi_n^{(n-1)}(s) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_n \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{n-1} \end{bmatrix}. \tag{150}$$

With $[\psi_{jk}]$ denoting the cofactor matrix of $[\phi_k^{(j)}(s)]_{jk}$, Cramer's rule yields

$$A_k = \frac{\alpha_0\psi_{0,k} + \dots + \alpha_{n-1}\psi_{n-1,k}}{W(s)}, \qquad k = 1, \dots, n, \tag{151}$$

and so

$$u(t) = \sum_{k=1}^{n} A_k\phi_k(t) = \sum_{k=1}^{n}\sum_{j=0}^{n-1} \frac{\alpha_j\psi_{j,k}}{W(s)}\phi_k(t) = \sum_{j=0}^{n-1} \frac{\alpha_j}{W(s)} \sum_{k=1}^{n} \phi_k(t)\psi_{j,k}, \tag{152}$$

which is (149) due to the Laplace expansion. □

15. Abel's formula

Let us prove Abel's formula for the n-th order equations. To this end, we will need to differentiate a determinant. According to the Leibniz expansion, the determinant of a matrix $A = [a_{ik}]$ can be written as

$$\det A = \sum_{\sigma} (-1)^{|\sigma|} a_{1\sigma_1} \cdots a_{n\sigma_n}, \tag{153}$$

where the sum is over all peermutations of the numbers $1, \dots, n$, and $|\sigma|$ is the parity of the permutation σ. Thinking of each element a_{ik} as a function of a scalar variable, we get

$$(\det A)' = \sum_{\sigma} (-1)^{|\sigma|} (a_{1\sigma_1}' \cdots a_{n\sigma_n} + a_{1\sigma_1} a_{2\sigma_2}' \cdots a_{n\sigma_n} + \dots + a_{1\sigma_1} \cdots a_{n\sigma_n}'), \tag{154}$$

which is to say, to differentiate a determinant, we simply take the derivative of each row at a time, and sum the resulting determinants:

$$(\det A)' = \begin{vmatrix} a_{11}' & a_{12}' & \dots & a_{1n}' \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21}' & a_{22}' & \dots & a_{2n}' \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1}' & a_{n2}' & \dots & a_{nn}' \end{vmatrix}. \tag{155}$$

An application of this to the Wronskian (147) yields

$$W' = \begin{vmatrix} \phi_1' & \dots & \phi_n' \\ \phi_1' & \dots & \phi_n' \\ \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix} + \dots + \begin{vmatrix} \phi_1 & \dots & \phi_n \\ \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix} + \dots + \begin{vmatrix} \phi_1 & \dots & \phi_n \\ \vdots & \dots & \vdots \\ \phi_1^{(n-2)} & \dots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \dots & \phi_n^{(n)} \end{vmatrix}. \quad (156)$$

Notice that with the exception of the last one, each determinant in the right hand side has two identical rows, and so

$$W' = \begin{vmatrix} \phi_1 & \dots & \phi_n \\ \vdots & \dots & \vdots \\ \phi_1^{(n-2)} & \dots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \dots & \phi_n^{(n)} \end{vmatrix}. \quad (157)$$

Now we use the fact that each ϕ_k satisfies the differential equation $p(D)u = 0$, that is, we substitute in the last row the expressions

$$\phi_k^{(n)} = -a_{n-1}\phi_k^{(n-1)} - \dots - a_1\phi_k' - a_0\phi_k, \quad (158)$$

to get

$$W' = -\begin{vmatrix} \phi_1 & \dots \\ \dots & \dots \\ \phi_1^{(n-2)} & \dots \\ a_{n-1}\phi_1^{(n-1)} & \dots \end{vmatrix} - \begin{vmatrix} \phi_1 & \dots \\ \dots & \dots \\ \phi_1^{(n-2)} & \dots \\ a_{n-2}\phi_1^{(n-2)} + \dots + a_1\phi_1' + a_0\phi_1 & \dots \end{vmatrix} = -a_{n-1}W, \quad (159)$$

where for reasons of space, only the first columns of the determinants are displayed. Thus we have *Abel's formula*

$$W(s) = W(0)e^{-a_{n-1}s} \qquad \text{for} \quad s \in \mathbb{R}. \quad (160)$$

As in the second order case, this immediately implies the following theorem.

Theorem 66. *Let $\phi_1, \dots, \phi_n$ be solutions of the differential equation $p(D)u = 0$. Then their Wronskian is either identically zero, or nonzero everywhere.*

Thus for solutions to differential equations of the type under consideration, we can talk about their Wronskian being simply zero or nonzero, without specifying where we evaluate the Wronskian.

16. Inhomogeneous equations

It is time to solve the inhomogenous equation

$$p(D)u = f. \quad (161)$$

All considerations and terminologies we had for the second order case naturally extend to the current situation.

Exercise 67. Prove the following.

(a) If u and v are solutions of (161), then the difference $w = u - v$ satisfies

$$p(D)w = 0. \quad (162)$$

(b) If v satsfies (161), and w satisfies (162), then $u = v + w$ satisfies (161).

(c) If $p(D)v_1 = f_1$ and $p(D)v_2 = f_2$, then the sum $v = v_1 + v_2$ satisfies $p(D)v = f_1 + f_2$.

Remark 68. Thus the general solution of (161) can be constructed as follows.

- Find a single *particular solution* v of the inhomogeneous equation (161).

- Find the general solution w of the *complementary equation* (162). In particular, this solution will contain n arbitrary constants.
- Then the sum $u = v + w$ is the general solution of (161).

Since we already know how to find the general solution of the homogenous equation (162), the focus now is on finding a single particular solution of (161).

Example 69. Consider the equation

$$u''' + u = e^{2t}. \tag{163}$$

The characteristic roots are $\lambda_{1,2} = (1 \pm i\sqrt{3})/2$ and $\lambda_3 = -1$, and so the general solution of the complementary equation is

$$w(t) = Ae^{-t} + e^{t/2}(Be^{i\sqrt{3}t/2} + Ce^{-i\sqrt{3}t/2}). \tag{164}$$

We look for a particular solution to (163) in the form $v(t) = Ge^{2t}$ and find $G = 1/9$. Thus the general solution of (163) is

$$u(t) = \frac{e^{2t}}{9} + Ae^{-t} + e^{t/2}(Be^{i\sqrt{3}t/2} + Ce^{-i\sqrt{3}t/2}), \tag{165}$$

where A, B and C are arbitrary constants.

Exercise 70. Find the general solution of $x'''' - x = t - \cos t$.

We shall now extend the method of variation of constants to the n-th order case. Let $\{\phi_1, \dots, \phi_n\}$ be a fundamental set of solutions to the equation $p(D)w = 0$. Since the functions $\phi_1, \dots, \phi_n$ cannot vanish at the same time, any function u can be written as

$$u(t) = A_1(t)\phi_1(t) + \dots + A_n(t)\phi_n(t), \tag{166}$$

with some coefficients $A_1(t), \dots, A_n(t)$.

Working formally, we compute

$$u' = A_1'\phi_1 + A_1\phi_1' + \dots + A_n'\phi_n + A_n\phi_n' = A_1\phi_1' + \dots + A_n\phi_n', \tag{167}$$

provided that

$$A_1'\phi_1 + \dots + A_n'\phi_n = 0. \tag{168}$$

We continue this trend, and assume

$$\begin{aligned} A_1'\phi_1' + \dots + A_n'\phi_n' &= 0, \\ &\dots \\ A_1'\phi_1^{(n-2)} + \dots + A_n'\phi_n^{(n-2)} &= 0, \end{aligned} \tag{169}$$

to get

$$\begin{aligned} u'' &= A_1'\phi_1' + A_1\phi_1'' + \dots + A_n'\phi_n' + A_n\phi_n'' = A_1\phi_1'' + \dots + A_n\phi_n'', \\ &\dots \\ u^{(n-1)} &= A_1'\phi_1^{(n-2)} + A_1\phi_1^{(n-1)} + \dots + A_n'\phi_n^{(n-2)} + A_n\phi_n^{(n-1)} = A_1\phi_1'' + \dots + A_n\phi_n'', \\ u^{(n)} &= A_1'\phi_1^{(n-1)} + A_1\phi_1^{(n)} + \dots + A_n'\phi_n^{(n-1)} + A_n\phi_n^{(n)}, \end{aligned} \tag{170}$$

Thus in order to have $p(D)u = f$, we need

$$A_1'\phi_1^{(n-1)} + \dots + A_n'\phi_n^{(n-1)} = f. \tag{171}$$

To summerize, if

$$\begin{aligned} A_1'\phi_1 + \ldots + A_n'\phi_n &= 0, \\ &\cdots \\ A_1'\phi_1^{(n-2)} + \ldots + A_n'\phi_n^{(n-2)} &= 0, \\ A_1'\phi_1^{(n-1)} + \ldots + A_n'\phi_n^{(n-1)} &= f, \end{aligned} \tag{172}$$

then $u = A_1\phi_1 + \ldots + A_n\phi_n$ would solve $p(D)u = f$. The system (172) can be solved for $A_1', \ldots, A_n'$ since $W[\phi_1, \ldots, \phi_n] \neq 0$. We make this argument precise in the following theorem.

Theorem 71. *Let $f \in C[a,b)$ with $-\infty < a < b \leq \infty$, and let u be a function n times differentiable in (a,b), with $u^{(n-1)} \in C[a,b)$ and*

$$p(D)u = f \quad \text{in } (a,b). \tag{173}$$

Then u is given by

$$u(t) = w(t) + \int_a^t \frac{f(s)}{W(s)} \begin{vmatrix} \phi_1(s) & \ldots & \phi_n(s) \\ \ldots & \ldots & \ldots \\ \phi_1^{(n-2)}(s) & \ldots & \phi_n^{(n-2)}(s) \\ \phi_1(t) & \ldots & \phi_n(t) \end{vmatrix} ds, \qquad t \in [a,b), \tag{174}$$

where w is the solution of the homogeneous initial value problem

$$\begin{cases} p(D)w = 0, \\ w(a) = u(a), \\ \ldots \\ w^{(n-1)}(a) = u^{(n-1)}(a). \end{cases} \tag{175}$$

Proof. For $t \in [a,b)$, let

$$A_k(t) = \int_a^t \frac{f(s)\psi_{n-1,k}(s)}{W(s)} ds, \qquad k = 1, \ldots, n, \tag{176}$$

where $[\psi_{j,k}(s)]_{j,k}$ is the cofactor matrix of $[\phi_k^{(j)}(s)]_{j,k}$. Note that the rows are numbered by $j = 0, \ldots, n-1$ (and the columns are by $k = 1, \ldots, n$), and so the element $\psi_{n-1,k}(s)$ is in the last row of the cofactor matrix. Then according to the fundamental theorem of calculus, A_k is differentiable in (a,b), and $A_k \in C[a,b)$, with

$$A_k' = \frac{f\psi_{n-1,k}}{W} \qquad \text{in} \quad (a,b). \tag{177}$$

Cramer's rule then implies that (172) is satisfied in (a,b). Now let

$$v(t) = \sum_{k=1}^n A_k(t)\phi_k(t) = \int_a^t \frac{f(s)}{W(s)} \sum_{k=1}^n \phi_k(t)\psi_{n-1,k}(s) ds, \tag{178}$$

for $t \in [a,b)$, which is the integral term in (174). We have $v \in C[a,b)$ and v is differentiable in (a,b) with

$$v' = \sum_{k=1}^n A_k'\phi_k + \sum_{k=1}^n A_k\phi_k' = \sum_{k=1}^n A_k\phi_k' \qquad \text{in} \quad (a,b). \tag{179}$$

Again, we have $v' \in C[a,b)$ and v' is differentiable in (a,b) with

$$v'' = \sum_{k=1}^n A_k'\phi_k' + \sum_{k=1}^n A_k\phi_k'' = \sum_{k=1}^n A_k\phi_k'' \qquad \text{in} \quad (a,b). \tag{180}$$

This reasoning can be repeated, until we get

$$v^{(n-1)} = \sum_{k=1}^{n} A_k' \phi_k^{(n-2)} + \sum_{k=1}^{n} A_k \phi_k^{(n-1)} = \sum_{k=1}^{n} A_k \phi_k^{(n-1)} \qquad \text{in} \quad (a,b), \tag{181}$$

and moreover, we have $v^{(n-1)} \in C[a,b)$ and $v^{(n-1)}$ is differentiable in (a,b) with

$$v^{(n)} = \sum_{k=1}^{n} A_k' \phi_k^{(n-1)} + \sum_{k=1}^{n} A_k \phi_k^{(n)} = f + \sum_{k=1}^{n} A_k \phi_k^{(n)} \qquad \text{in} \quad (a,b). \tag{182}$$

As $p(D)\phi_k = 0$ for all k, we infer

$$p(D)v = f \qquad \text{in} \quad (a,b). \tag{183}$$

Hence the difference $w = u - v$ satisfies $p(D)w = 0$ in (a,b). On the other hand, since $A_k(a) = 0$ for all k, we have $v^{(j)}(a) = 0$ for all $j = 0, \dots, n-1$, and so $w^{(j)}(a) = u^{(j)}(a)$ for all $j = 0, \dots, n-1$. By uniqueness, there is exactly one such function. The proof is complete. □

17. First order homogeneous systems

The entire theory of constant coefficient linear equations can be subsumed under the theory of constant coefficient *first order systems*. The homogeneous version of such a system is

$$x' = Ax, \tag{184}$$

where the unknown is a vector function $x(t) \in \mathbb{R}^n$, and A is a given $n \times n$ matrix with real entries. It is also possible to consider complex matrices, in which case the unknown would have to be a complex vector function $x(t) \in \mathbb{C}^n$.

Example 72. The second order equation

$$u'' + pu' + qu = 0, \tag{185}$$

can be transformed into a first order system by introducing the new variable $v = u'$:

$$\begin{cases} u' = v \\ v' = -pv - qu \end{cases} \tag{186}$$

or

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{187}$$

Exercise 73. Write the n-th order equation

$$u^{(n)} + a_{n-1} u^{(n-1)} + \ldots + a_1 u' + a_0 u = 0, \tag{188}$$

as a first order system.

Exercise 74. Write the first order *complex* equation

$$u' = \ell u, \qquad (\ell \in \mathbb{C}), \tag{189}$$

as a first order *real* system.

Let us look for a solution of $x' = Ax$ in the form

$$x(t) = x_0 + t x_1 + t^2 x_2 + t^3 x_3 + \ldots, \tag{190}$$

where $x_0, x_1, \ldots \in \mathbb{C}^n$ are fixed vectors that will be adjusted shortly. Taking the derivative term by term and equating it with Ax, we get

$$x_1 + 2t x_2 + 3t^2 x_3 + \ldots = Ax_0 + tAx_1 + t^2 Ax_2 + \ldots. \tag{191}$$

Since this must hold for arbitrary t, the coefficients of the powers of t on both sides should match term by term:

$$\begin{aligned} x_1 &= Ax_0, \\ x_2 &= \frac{1}{2}Ax_1 = \frac{1}{2}A^2x_0, \\ x_3 &= \frac{1}{3}Ax_2 = \frac{1}{2\cdot 3}A^3x_0, \\ x_4 &= \frac{1}{4}Ax_3 = \frac{1}{2\cdot 3\cdot 4}A^4x_0, \quad \dots \end{aligned} \tag{192}$$

yielding the general formula

$$x_m = \frac{1}{m!}A^m x_0, \qquad m = 0, 1, \dots. \tag{193}$$

This gives us the following candidate solution

$$\begin{aligned} x(t) &= x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \frac{t^3}{3!}A^3x_0 + \dots \\ &= \big(I + tA + \frac{(tA)^2}{2} + \frac{(tA)^3}{3!} + \dots\big)x_0. \end{aligned} \tag{194}$$

Notice that the series between the brackets resembles the Maclaurin expansion of the exponential function. We call it the *exponential of* tA and write

$$e^{tA} = \exp(tA) = \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} = I + tA + \frac{(tA)^2}{2} + \frac{(tA)^3}{3!} + \dots. \tag{195}$$

We can think of the series in the right hand side as a series of $n \times n$ matrices, or equivalently, a collection of n^2 individual scalar series.

Theorem 75. *For any given $n \times n$ matrix A, each individual series in (195) converges absolutely and uniformly in any bounded interval of $\mathbb{R}$, and*

$$\frac{d}{dt}e^{tA} = e^{tA}A = Ae^{tA}, \qquad t \in \mathbb{R}. \tag{196}$$

Moreover, any vector function that is differentiable everywhere and that satisfies $x' = Ax$ everywhere in $\mathbb{R}$ has the representation $x(t) = e^{tA}x(0)$.

Proof. Let us introduce the notation

$$|A|_\infty = \max_{j,k} |a_{jk}|, \tag{197}$$

where a_{jk} are the elements of A, and suppose that $M = |A|_\infty$, that is, the entries of A are bounded in absolute value by M. Then it is easy to see that

$$|A^m|_\infty \le M^m n^{m-1}, \tag{198}$$

and so

$$\sum_{m=0}^{\infty} \frac{|(tA)^m|_\infty}{m!} \le \sum_{m=0}^{\infty} \frac{|t|^m M^m n^{m-1}}{m!} = \frac{e^{Mn|t|}}{n}. \tag{199}$$

Thus by the Weierstrass M-test we conclude that each individual entry in the matrix series converges absolutely and uniformly in any bounded interval of $\mathbb{R}$. Since this is a power series in t, we can differentiate term by term, and get

$$\frac{d}{dt}e^{tA} = \frac{d}{dt}\sum_{m=0}^{\infty} \frac{t^m A^m}{m!} = \sum_{m=1}^{\infty} \frac{t^{m-1}A^m}{(m-1)!} = \sum_{m=0}^{\infty} \frac{t^m A^{m+1}}{m!} = e^{tA}A = Ae^{tA}, \tag{200}$$

for $t \in \mathbb{R}$. The remaining (uniqueness) claim is immediate because

$$(e^{-tA}x(t))' = -e^{-tA}Ax(t) + e^{-tA}x'(t) = e^{-tA}(-Ax(t) + x'(t)) = 0, \tag{201}$$

for all t, implying that $e^{-tA}x(t) = \text{const}$. The proof is complete. □

Example 76. We know that the solution to the initial value problem

$$\begin{cases} x'' + x = 0, \\ x(0) = \alpha, \\ x'(0) = \beta, \end{cases} \tag{202}$$

is $x(t) = \alpha\cos t + \beta \sin t$. With $y = x'$, this can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{203}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \alpha\cos t + \beta\sin t \\ -\alpha\sin t + \beta\cos t \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{204}$$

This means that

$$e^{tJ} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \qquad \text{with} \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{205}$$

Exercise 77. Let A and B be square matrices with $AB = BA$. Prove the following.
(a) $(e^A)^{-1} = e^{-A}$.
(b) $e^A B = Be^A$.
(c) $e^{A+B} = e^A e^B$.

Hint: (a) Differentiate $F(t) = e^{-tA}e^{tA}$ with respect to t. (b) Work directly with the power series (195). (c) Differentiate $F(t) = e^{t(A+B)}e^{-tA}e^{-tB}$.

18. Matrix exponentials

In this section, we will see ways to explicitly compute the matrix exponential. The simplest case is when the matrix is diagonal: Since

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}^m = \begin{bmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^m \end{bmatrix}, \tag{206}$$

it is immediate from (195) that

$$e^{t\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}, \qquad \text{when} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}. \tag{207}$$

Next, if A is *diagonalizable*, i.e., if $A = S\Lambda S^{-1}$ with some diagonal matrix Λ, then

$$\begin{aligned} e^{tA} &= I + tS\Lambda S^{-1} + \frac{t^2}{2}S\Lambda S^{-1}S\Lambda S^{-1} + \frac{t^3}{3!}S\Lambda S^{-1}S\Lambda S^{-1}S\Lambda S^{-1} + \dots \\ &= S\big(I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \frac{t^2}{3!}\Lambda^3 + \dots\big)S^{-1} \\ &= Se^{t\Lambda}S^{-1}. \end{aligned} \tag{208}$$

Remark 78. Thus if one needs to compute e^{tA}, it is a good idea to check if A is diagonalizable by finding its eigenvalues and eigenvectors. If A has a complete set of eigenvectors, say, $v_1, \dots, v_n$ with the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then we can rewrite

$$Av_k = \lambda_k v_k, \qquad k = 1, \dots, n, \tag{209}$$

in the form

$$AV = V\Lambda, \qquad \text{or} \qquad A = V\Lambda V^{-1}, \tag{210}$$

where Λ is the diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ on its diagonal, and V is the square matrix with $v_1, \dots, v_n$ as its columns.

Note that V^{-1} times an eigenvector v_k equals the k-th standard basis vector e_k, and so

$$e^{tA}v_k = Ve^{t\Lambda}V^{-1}v_k = Ve^{t\Lambda}e_k = Ve^{\lambda_k t}e_k = e^{\lambda_k t}v_k. \tag{211}$$

By linearity, we get

$$e^{tA}\sum_{k=1}^{n}\alpha_k v_k = \sum_{k=1}^{n} e^{\lambda_k t}\alpha_k v_k, \tag{212}$$

which shows that the components of the initial vector $x(0) = \sum_k \alpha_k v_k$ simply scale as $e^{\lambda_k t}\alpha_k$ independently of each other under the evolution prescribed by the equation $x' = Ax$.

Remark 79. If A is symmetric (i.e., $A = A^\top$), then A is diagonalizable by an orthogonal transformation, that is, there is an ortogonal V and a diagonal Λ such that $A = V\Lambda V^\top$. If A is Hermitian (i.e., $A = A^\dagger$), then A is diagonalizable by a unitary transformation, that is, there is a unitary U and a diagonal Λ such that $A = U\Lambda U^\dagger$.

Example 80. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{213}$$

Its egenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$ with the corresponding eigenvectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$. Hence the matrix V is

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \text{with} \quad V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{214}$$

giving

$$e^{tA} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} e^{3t}+e^{-t} & e^{3t}-e^{-t} \\ e^{3t}-e^{-t} & e^{3t}+e^{-t} \end{bmatrix}. \tag{215}$$

What if the matrix is *not* diagonalizable? Let us start with a couple of simple examples.

Example 81. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 2I + N, \qquad \text{where} \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{216}$$

One can easily find that the characteristic equation $(\lambda-2)^2 = 0$ has a double root at $\lambda = 2$, and that the only eigenvector corresponding to $\lambda = 2$ is $e_1 = (1, 0)$. Thus A is non-diagonalizable. We say that the eigenvalue $\lambda = 2$ is *defective*.

Nevertheless, we can compute e^{tA} without difficulty. Since $N^2 = 0$, we note that

$$A^m = (2I + N)^m = 2^m I + m2^{m-1}N, \qquad m = 1, 2, \dots, \tag{217}$$

yielding

$$\begin{aligned} e^{tA} &= \sum_{m=0}^{\infty} \frac{t^m}{m!}(2I+N)^m = \sum_{m=0}^{\infty} \frac{t^m}{m!} 2^m I + \sum_{m=1}^{\infty} \frac{t^m}{(m-1)!} 2^{m-1} N \\ &= e^{2t} I + t e^{2t} N = e^{2t}(I + tN) \\ &= e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}. \end{aligned} \tag{218}$$

The action of e^{tA} on e_1 and e_2 are

$$e^{tA} e_1 = e^{2t} e_1, \qquad e^{tA} e_2 = e^{2t} e_2 + t e^{2t} e_1. \tag{219}$$

This shows that even though e_1 evolves by pure scaling, the evolution of e_2 "spills" into e_1. In fact, apart from e_1 there is no other vector that evolves by pure scaling.

Example 82. Consider the matrix

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I + N, \qquad \text{where} \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{220}$$

As in the preceding example, we note $N^3 = 0$ and hence

$$A^m = (\lambda I + N)^m = \lambda^m I + m\lambda^{m-1} N + \frac{m(m-1)}{2} \lambda^{m-2} N^2, \tag{221}$$

for $m = 2, 3, \ldots$. This gives

$$\begin{aligned} e^{tA} &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \lambda^m I + \sum_{m=1}^{\infty} \frac{t^m}{(m-1)!} \lambda^{m-1} N + \sum_{m=2}^{\infty} \frac{t^m}{2(m-2)!} \lambda^{m-2} N^2 \\ &= e^{\lambda t} I + t e^{\lambda t} N + \frac{t^2}{2} e^{\lambda t} N^2 = e^{\lambda t}(I + tN + \frac{t^2}{2} N^2), \end{aligned} \tag{222}$$

or more explicitly

$$e^{tA} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}. \tag{223}$$

The evolution of the standard basis is

$$e^{tA} e_1 = e^{2\lambda t} e_1, \qquad e^{tA} e_2 = e^{\lambda t} e_2 + t e^{\lambda t} e_1, \qquad e^{tA} e_2 = e^{\lambda t} e_3 + t e^{\lambda t} e_2 + \frac{1}{2} t^2 e^{\lambda t} e_1. \tag{224}$$

Thus e_2 spills into e_1 but *not* into e_3, and e_3 spills into both e_1 and e_2.

Exercise 83. Compute e^{tA} where the matrix A is given by

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{225}$$

Remark 84. Any square matrx A has the so-called *Jordan decomposiion*

$$A = VJV^{-1}, \tag{226}$$

where J is a block diagonal matrix whose diagonal consists of blocks of the form

$$J_k = \begin{bmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_k \end{bmatrix}. \tag{227}$$

We write $J = \operatorname{diag}(J_1, \dots, J_m)$ or $J = J_1 \oplus J_2 \oplus \dots \oplus J_m$. The matrix J is called the *Jordan normal form* of A, and J_k are the *Jordan blocks* of A. If J_k is at least 2×2, then we say that the eigenvalue λ_k is *defective*. Denoting by d_k the size of J_k, it is easy to see that

$$e^{tJ_k} = e^{\lambda_k t}\big(I + tN + \dots + \frac{t^{d_k-1}}{(d_k-1)!}N^{d_k-1}\big), \tag{228}$$

where $N = J_k - \lambda_k I$ is the $d_k \times d_k$ matrix with its first superdiagonal consising of 1's, or

$$e^{tJ_k} = e^{\lambda_k t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \dots & \frac{1}{(d_k-1)!}t^{d_k-1} \\ 0 & 1 & t & \dots & \frac{1}{(d_k-2)!}t^{d_k-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{229}$$

Thus we have

$$e^{tA} = Ve^{tJ}V^{-1} = V(e^{tJ_1} \oplus e^{tJ_2} \oplus \dots \oplus e^{tJ_m})V^{-1}. \tag{230}$$

Example 85. For the block diagonal matrix

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \tag{231}$$

we have

$$e^{tJ} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} & 0 & 0 \\ 0 & 0 & e^{\lambda t} & 0 & 0 \\ 0 & 0 & 0 & e^{\mu t} & te^{\mu t} \\ 0 & 0 & 0 & 0 & e^{\mu t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \oplus e^{\mu t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{232}$$

Exercise 86. Compute e^{tA} where the matrix A is given by

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}. \tag{233}$$

19. Stability

Now that we know how to compute the matrix exponential e^{tA}, we can study the stability of the system $x' = Ax$. First, it is quite obvious that this system is strictly stable if and only if every eigenvalue of A has strictly negative real part. Second, if an eigenvalue with zero real part has a Jordan block larger than 1×1, we would have solutions such as $te^{\lambda t}$ with λ purely imaginary, and hence would not be stable. Thus in order for the system be stable, purely imaginary (or zero) eigenvalues cannot be defective. Let us state this result as a theorem.

Theorem 87. *The system $x' = Ax$ is strictly stable if and only if every eigenvalue of A has strictly negative real part. Moreover, $x' = Ax$ is stable if and only if every eigenvalue of A has nonpositive real part, and all its purely imaginary (or zero) eigenvalues are non-defective.*

Proof. Any function of the form $t^k e^{\lambda t}$ with $\operatorname{Re}(\lambda) < 0$ decays to 0 as $t \to \infty$, and $e^{\lambda t}$ with $\operatorname{Re}(\lambda) \geq 0$ doesn't decay as $t \to \infty$. Thus the first claim of the theorem is established.

Similarly, any function of the form $t^k e^{\lambda t}$ with $\operatorname{Re}(\lambda) < 0$ or of the form $e^{\lambda t}$ with $\operatorname{Re}(\lambda) = 0$ stays bounded as $t \to \infty$. On the other hand, neither $e^{\lambda t}$ with $\operatorname{Re}(\lambda) > 0$ nor $te^{\lambda t}$ with $\operatorname{Re}(\lambda) = 0$ stays bounded as $t \to \infty$. The proof is complete. □

20. Fundamental matrices

We have established that the unique solution to the initial value problem

$$\begin{cases} x' = Ax, \\ x(0) = \alpha, \end{cases} \tag{234}$$

is given by

$$x(t) = e^{tA}\alpha, \qquad t \in \mathbb{R}. \tag{235}$$

Taking the initial condition α as the columns of the identity matrix, that is, taking $x(0) = e_k$ with $k = 1, \dots, n$, we see that the matrix function $E(t) = e^{tA}$ is the unique solution of the matrix initial value problem

$$\begin{cases} E'(t) = AE(t), \\ E(0) = I. \end{cases} \tag{236}$$

More generally, if $\Phi(t)$ is a matrix function satisfying $\Phi' = A\Phi$, then the vector function $x(t) = \Phi(t)\beta$ for any fixed vector β satisfies

$$x' = \Phi'\beta = A\Phi\beta = Ax, \tag{237}$$

and

$$x(0) = \Phi(0)\beta. \tag{238}$$

Thus in order to solve the initial value problem (234), it suffices to invert the matrix $\Phi(0)$:

$$\beta = [\Phi(0)]^{-1}\alpha, \qquad \text{or} \qquad x(t) = \Phi(t)[\Phi(0)]^{-1}\alpha. \tag{239}$$

Definition 88. A *fundamental matrix* of the system $x' = Ax$ is a matrix function $\Phi(t)$ satisyfing $\Phi' = A\Phi$ and $\det \Phi(0) \neq 0$.

Remark 89. If $\Phi(t)$ is any matrix function satisfying $\Phi' = A\Phi$, then a columnwise application of (235) gives $\Phi(t) = e^{tA}\Phi(0)$. Hence we have

$$[\Phi(t)]^{-1} = [\Phi(0)]^{-1}e^{-tA}, \tag{240}$$

meaning that $\Phi(0)$ is invertible if and only if $\Phi(t)$ is invertible, for any $t \in \mathbb{R}$. Thus similarly to fundamental sets for ODEs, instead of checking $\det \Phi(0) \neq 0$ one can check the property $\det \Phi(t) \neq 0$ at any point t, provided of course that $\Phi(t)$ satisfies $\Phi' = A\Phi$.

Exercise 90. Show that if $\Phi(t)$ is a fundamental matrix of $x' = Ax$, then the unique solution of the initial value problem

$$\begin{cases} x' = Ax, \\ x(s) = \alpha, \end{cases} \tag{241}$$

is given by

$$x(t) = \Phi(t)\Phi(s)^{-1}\alpha, \qquad t \in \mathbb{R}. \tag{242}$$

21. Liouville's formula

In Section 15, we derived the analogue of the property $\det\Phi(0) \neq 0 \Leftrightarrow \det\Phi(t) \neq 0$ for fundamental sets of ODEs by proving the more quantitative *Abel formula*. For linear systems this corresponds to the following *Liouville formula*.

Theorem 91. *If $\Phi(t)$ is a matrix function satisfying $\Phi' = A\Phi$, then we have*

$$(\det\Phi)' = \mathrm{tr}A \det\Phi, \tag{243}$$

where $\mathrm{tr}A$ *is the trace of A, i.e., the sum of the diagonal entries of A.*

Proof. Denote the j-th row of Φ by ϕ_j. Then $\Phi' = A\Phi$ can be written row by row as

$$\phi_j' = \sum_{k=1}^{n} a_{jk}\phi_k, \qquad j = 1, \dots, n, \tag{244}$$

where a_{jk} are the elements of the matrix A. Now we compute $(\det\Phi)'$ as

$$\begin{aligned}\frac{d}{dt}\begin{vmatrix}\phi_1\\ \phi_2\\ \dots\\ \phi_n\end{vmatrix} &= \begin{vmatrix}\phi_1'\\ \phi_2\\ \dots\\ \phi_n\end{vmatrix} + \begin{vmatrix}\phi_1\\ \phi_2'\\ \dots\\ \phi_n\end{vmatrix} + \dots + \begin{vmatrix}\phi_1\\ \phi_2\\ \dots\\ \phi_n'\end{vmatrix} = \begin{vmatrix}a_{11}\phi_1 + a_{12}\phi_2 + \dots + a_{1n}\phi_n\\ \phi_2\\ \dots\\ \phi_n\end{vmatrix}\\ &+ \begin{vmatrix}\phi_1\\ a_{21}\phi_1 + a_{22}\phi_2 + \dots + a_{2n}\phi_n\\ \dots\\ \phi_n\end{vmatrix} + \dots + \begin{vmatrix}\phi_1\\ \phi_2\\ \dots\\ a_{n1}\phi_1 + a_{n2}\phi_2 + \dots + a_{nn}\phi_n\end{vmatrix}\\ &= \begin{vmatrix}a_{11}\phi_1\\ \phi_2\\ \dots\\ \phi_n\end{vmatrix} + \begin{vmatrix}\phi_1\\ a_{22}\phi_2\\ \dots\\ \phi_n\end{vmatrix} + \dots + \begin{vmatrix}\phi_1\\ \phi_2\\ \dots\\ a_{nn}\phi_n\end{vmatrix} = (a_{11} + a_{22} + \dots + a_{nn})\begin{vmatrix}\phi_1\\ \phi_2\\ \dots\\ \phi_n\end{vmatrix},\end{aligned} \tag{245}$$

which concludes the proof. □

22. Duhamel's formula

Consider the inhomogeneous system

$$x' = Ax + f, \tag{246}$$

where $f(t)$ is a vector function continuous in $[a, b)$. Assuming that $x(t)$ is also continuous in $[a, b)$ and differentiable in (a, b), we have

$$(e^{-tA}x(t))' = -e^{-tA}Ax(t) + e^{-tA}x'(t) = e^{-tA}(-Ax(t) + x'(t)) = e^{-tA}f(t), \tag{247}$$

and so

$$e^{-tA}x(t) = x(a) + \int_a^t e^{-sA}f(s)ds, \tag{248}$$

or

$$x(t) = e^{tA}x(a) + \int_a^t e^{(t-s)A}f(s)ds, \qquad t \in [a, b). \tag{249}$$

This is *Duhamel's formula* for constant coefficient linear systems.

Exercise 92. Let A be a square matrix, let $f(t)$ be a vector function continuous in $[a, b)$, and let α be a vector. Show that

$$x(t) = e^{tA}\alpha + \int_a^t e^{(t-s)A}f(s)ds, \qquad t \in [a, b), \tag{250}$$

is a solution to the initial value problem

$$\begin{cases} x' = Ax + f \quad \text{in } (a,b), \\ x(a) = \alpha. \end{cases} \tag{251}$$

Exercise 93. Let $\Phi(t)$ be a fundamental matrix of the system $x' = Ax$, and let $f(t)$ be a vector function continuous in $[a,b)$. Show that any solution of $x' = Ax + f$ that is continuous in $[a,b)$ and differentiable in (a,b) has the representation

$$x(t) = \Phi(t)\Phi(a)^{-1}x(a) + \int_a^t \Phi(t)\Phi(s)^{-1}f(s)ds, \qquad t \in [a,b). \tag{252}$$